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The vacuum energy for two cylinders with one increasing in size

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Abstract
We consider the vacuum energy for the configuration of two cylinders and obtain its asymptotic expansion if the radius of one of these cylinders becomes large while the radius of the other one and their separation are kept fixed. We calculate explicitly the next-to-leading order correction to the vacuum energy for the radius of the other cylinder becoming large or small.

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1. Introduction

During the past years, remarkable progress was made in the field of the Casimir effect [1] and in the calculation of the vacuum interaction energy and the Casimir force for separated bodies of nontrivial shape. It turned out that it is possible to write down a representation of this vacuum energy in terms of a functional determinant which does not contain ultraviolet divergences. The first applications had been made for the configuration of two parallel cylinders and two spheres [2, 3]. From here, the corresponding expressions for a cylinder or a sphere in front of a plane follow for symmetry reasons. An alternative approach by [4] should be mentioned which finally delivers the same results. There are some precursors, [5–7], and a large number of followers, [8–15].

The approach using functional determinants allows for a direct numerical evaluation since all involved summations and integrations do converge. Especially for large separation between the interacting bodies, the convergence is rapid. It is also easy to obtain an asymptotic expansion in this limit. It corresponds to a dipole approximation where only the lowest orbital momenta are involved. In the opposite limit of small separation, the situation is more complicated. Here arbitrarily high orbital momenta give a significant contribution, and reliable numerical results are hard to obtain. However, the asymptotic expansion for small separation

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can be calculated analytically. For a cylinder in front of a plane, this was done in [16] and for a sphere in [17]. As a result, for these configurations, the proximity force approximation (PFA) was re-confirmed and the first correction beyond was calculated. These results were partly confirmed by numerical approaches. First, we mention the remarkable world line method [18–20], which confirmed the above-mentioned results for the case of Dirichlet boundary conditions. These results were also confirmed by extrapolation of the numerical evaluation of the functional determinant from finite to small separation [21, 22]. In line with these, it should be mentioned that for Neumann boundary conditions the numerical results are less reliable, and a satisfactory agreement with the analytical results could not be reached so far.

In this paper, we consider analytically a further limiting case, namely a cylinder A of fixed radius R_A at finite separation d from a second cylinder B (see figure 1), whose radius R_B becomes large, $R_B \to \infty$, while keeping d fixed. In the limiting case we reproduce, of course, the result for a cylinder in front of a plane. We derive the general expressions for the first two corrections for large R_B and calculate the first-order correction explicitly.

It should be mentioned that the limit of one cylinder becoming large cannot be obtained by PFA since the separation between the two cylinders remains finite. As well, this limit cannot be obtained by a dipole approximation since arbitrarily high orders of the orbital momenta related to the cylinder B contribute. This can be seen below on the hand of the approximation of the kernel $K^{-1}_{BB}(\psi, \psi')$, equation (31).

Finally, it should be mentioned that some time ago, there was a proposal to measure the Casimir force between a cylinder and a plane [23]. This paper could be helpful for the calculation of the corresponding forces.

This paper is organized as follows. In the following section, we re-derive the formulae for the vacuum energy in the presence of two cylinders. We need these formulae in order to introduce the necessary notations adopted to the needs of section 3, where we consider the limit $R_B \to \infty$. Finally, section 4 contains some discussion and conclusions. Throughout this paper we use units with $\hbar = c = 1$.

2. The basic formulae for two cylinders

In this section, we display the formulae for the vacuum energy of a massless scalar field obeying Dirichlet boundary conditions on two parallel cylinders. We follow the method used in [2, 16]; see also in [1]. The configuration is shown in figure 1. The vacuum energy can be written in the form of a trace of the logarithm,

$$E = \frac{1}{4\pi} \int_0^\infty \mathrm{d}\omega\, \omega\, \mathrm{Tr} \ln(\delta(\varphi - \varphi') - M(\varphi, \varphi')), \tag{1}$$

with

$$M(\varphi, \varphi') = K^{-1}_{AA}(\varphi, \varphi'') K_{AB}(\varphi'', \psi) K^{-1}_{BB}(\psi, \psi') K_{BA}(\psi', \varphi'), \tag{2}$$

where integration over doubly occurring angles is assumed and the trace in (1) is over the angles φ and φ'. In these formulae, the kernels $K_{AB}(\varphi, \psi)$ are the projections of the free space propagator

$$\Delta_\omega(\vec{x}, \vec{x}') = \int \frac{\mathrm{d}\vec{k}}{(2\pi)^2} \frac{\mathrm{e}^{\mathrm{i}\vec{k}(\vec{x}-\vec{x}')}}{\omega^2 + k^2}, \tag{3}$$

which is taken in the Fourier representation with respect to the translational invariant directions x_0 and x_3, onto the surfaces of the cylinders. The vectors $\vec{x}$ and $\vec{x}'$ are in the (x_1, x_2)-plane

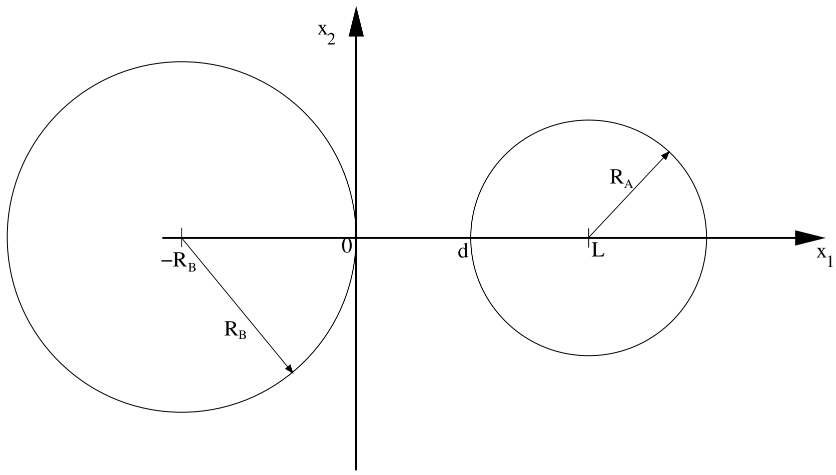

Figure 1. The configuration of two cylinders.

and $\vec{k} = (k_1, k_2)$ is the corresponding momentum. For the cylinders we use the following parameterizations,

$$A: \quad \vec{x}^A(\varphi) = \begin{pmatrix} L + R_A \cos\varphi \\ R_A \sin\varphi \end{pmatrix}, \qquad B: \quad \vec{x}^B(\varphi) = \begin{pmatrix} R_B(-1 + \cos\varphi) \\ R_B \sin\varphi \end{pmatrix}, \tag{4}$$

so that we have

$$K_{AB}(\varphi, \psi) = \Delta_\omega(\vec{x}^A(\varphi), \vec{x}^B(\psi)) \tag{5}$$

and accordingly with $A \leftrightarrow B$. The inverse of a kernel is taken in the sense of an operation on the surface of the cylinder, i.e.,

$$\int \mathrm{d}\varphi'' K_{AA}^{-1}(\varphi, \varphi'') K_{AA}(\varphi'', \varphi') = \delta(\varphi - \varphi') \tag{6}$$

must hold. In equations (1) and (2) the integration over the corresponding angles is assumed. The interval for all angular integrations is $\varphi \in [0, 2\pi]$. As well, the trace is to be taken in this sense, for instance,

$$\mathrm{Tr}\, M(\varphi, \varphi') = \int_0^{2\pi} \mathrm{d}\varphi\, M(\varphi, \varphi). \tag{7}$$

In this way, all quantities entering the representation (1) of the vacuum energy are defined. However, in order to work with explicit formulae one needs to change this representation by introducing an appropriate basis in which the inverse kernels become diagonal. According to the geometry of the considered problem, we use the basis

$$|l\rangle = \frac{\mathrm{e}^{\mathrm{i}l\varphi}}{\sqrt{2\pi}} \tag{8}$$

and the notations

$$\langle l|K_{CC'}|l'\rangle \equiv K_{CC';l,l'} = \int \mathrm{d}\varphi\, \mathrm{d}\varphi' \,\frac{\mathrm{e}^{-\mathrm{i}l\varphi + \mathrm{i}l'\varphi'}}{2\pi} K_{CC'}(\varphi, \varphi'), \tag{9}$$

where C and C' stand for any A or B. On this basis, all quantities become infinite dimensional matrices in the indices (l, l') $(l = -\infty, \dots, \infty)$, and the vacuum energy can be rewritten in the form

$$E = \frac{1}{4\pi}\int_0^\infty \mathrm{d}\omega\,\omega\,\mathrm{Tr}\ln(\delta_{l,l'} - M_{l,l'}). \tag{10}$$

For the needs of the following section, it is useful to represent the matrix $M_{l,l'}$ by

$$M_{l,l'} = K^{-1}_{AA;l}\,N_{l,l'} \tag{11}$$

(the matrix $K^{-1}_{AA;l,l'}$ is diagonal; see equation (19)), where $N_{l.l'}$ can be written in coordinate space as

$$N_{l.l'} = \langle l|N(\varphi,\varphi')|l'\rangle \tag{12}$$

with

$$N(\varphi,\varphi') = \int \mathrm{d}\psi\,\mathrm{d}\psi'\;K_{AB}(\varphi,\psi)K^{-1}_{BB}(\psi,\psi')K_{BA}(\psi',\varphi') \tag{13}$$

or, in orbital momentum representation, as

$$N_{l,l'} = \sum_{l''} K_{AB;l,l''}K^{-1}_{BB;l''}K_{BA;l'',l'}. \tag{14}$$

In terms of these quantities, the logarithm in the formula (10) for the vacuum energy can be expanded and one comes to the completely explicit representation

$$E = \frac{1}{4\pi}\int_0^\infty \mathrm{d}\omega\,\omega\sum_{s=0}^{\infty}\frac{-1}{s+1}\sum_{l=-\infty}^{\infty}\sum_{l_1=-\infty}^{\infty}\cdots\sum_{l_s=-\infty}^{\infty} M_{l,l_1}M_{l_1,l_2}\dots M_{l_s,l}. \tag{15}$$

As already said, this is a finite expression, i.e., the integration and all summations in (15) do converge for any fixed values of the parameters R_A, R_B and L.

In the next step, we need to remind ourselves of the known explicit expressions for the matrices $K^{-1}_{AA;l}$ and $N_{l,l'}$. We start with $K_{AA;l,l'}$ which is in accordance with (9) and (5) given by

$$K_{AA;l,l'} = \int \mathrm{d}\varphi\,\mathrm{d}\varphi'\,\frac{\mathrm{e}^{-\mathrm{i}l\varphi+\mathrm{i}l'\varphi'}}{2\pi}\Delta_\omega(\vec{x}_A(\varphi),\vec{x}_A(\varphi')). \tag{16}$$

Using the expansion

$$\mathrm{e}^{\mathrm{i}z\cos\varphi} = \sum_{l=-\infty}^{\infty} i^l J_l(z)\,\mathrm{e}^{\mathrm{i}l\varphi} \tag{17}$$

of a plane wave into cylindrical ones, equation (16) can be rewritten as

$$\begin{aligned}K_{AA;l,l'} \equiv K_{AA;l}\,\delta_{l,l'} &= \int_0^\infty \frac{\mathrm{d}k\,k}{2\pi}\,J_l(kR_A)^2,\\ &= I_l(\omega R_A)K_l(\omega R_A).\end{aligned} \tag{18}$$

The last line follows from equation (6.541 1) in [24] and is in terms of the modified Bessel functions. This kernel is diagonal in the orbital momenta and can be inverted simply by

$$K^{-1}_{AA;l,l'} \equiv K^{-1}_{AA;l}\,\delta_{l,l'} = \frac{\delta_{l,l'}}{I_l(\omega R_A)K_l(\omega R_A)}. \tag{19}$$

The corresponding formulae for $K^{-1}_{BB;l,l'}$ follow by substituting the radius, $R_A \to R_B$.

The remaining matrices originate from free space Green's functions having their legs on different cylinders,

$$K_{AB;l,l'} = \int \mathrm{d}\varphi\,\mathrm{d}\varphi'\,\frac{\mathrm{e}^{-\mathrm{i}l\varphi+\mathrm{i}l'\varphi'}}{2\pi}\Delta_\omega(\vec{x}_A(\varphi),\vec{x}_B(\varphi')). \tag{20}$$

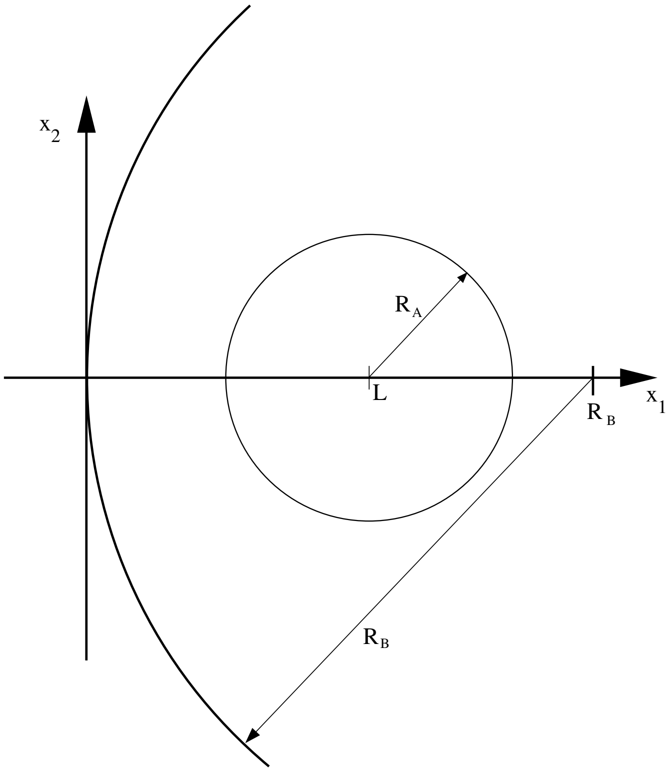

Figure 2. The configuration of two cylinders one inscribed into the other.

Using the expansion (17) three times and a corresponding formula generalizing equation (6.541 1) in [24] one comes to

$$K_{AB;l,l'} = (-1)^l I_l(\omega R_A) K_{l-l'}(\omega(L+R_B)) I_{l'}(\omega R_B). \tag{21}$$

This formula gives the transition from the cylinder A to the cylinder B if using the terminology of the transition formula approach. The reverse formula follows by spatial reflection of the plane $x_1 = (L+R_A)/2$,

$$K_{BA;l,l'} = (-1)^{l+l'} K_{AB;l',l}. \tag{22}$$

In a similar way, with the substitution

$$R_B \to -R_B \tag{23}$$

one comes to the formula

$$K_{AB;l,l'} = (-1)^l I_l(\omega R_A) I_{l-l'}(\omega(R_B-L)) K_{l'}(\omega R_B), \tag{24}$$

which corresponds to the configuration of the cylinder A inside the cylinder B; see figure 2.

With the above formulae, i.e., by inserting equation (18) and (19), (21) and (22) into (14), we get for (11)

$$M_{l,l'} = (-1)^{l+l'} \sum_{l''=-\infty}^{\infty} \frac{1}{K_l(\omega R_A)} I_{l-l''}(\omega(R_A-L)) \frac{I_{l''}(\omega R_B)}{K_{l''}(\omega R_B)}$$
$$I_{l''-l'}(\omega(R_B-L)) K_{l'}(\omega R_B). \tag{25}$$

Being inserted into (15) this formula delivers the known expression for the vacuum energy with Dirichlet boundary conditions on the two cylinders of radii R_A and R_B. As already mentioned, this formula allows for a direct numerical evaluation at fixed R_A, R_B and L.

3. One cylinder becoming large

In this section, we consider the vacuum energy (10) in the limit $R_B \to \infty$. In the leading order, we reproduce the corresponding expression for one cylinder in front of a plane. The next-to-leading order will then be the main result of this paper.

We start from representation (10) of the vacuum energy with $M_{l,l'}$ given by equation (11) and the inverse kernel $K^{-1}_{AA;l}$ given by equation (19). All quantities related to the cylinder B are contained in $N_{l,l'}$, equation (12). For the following, it is convenient to consider its coordinate space representation (13). The main step for considering the limit $R_B \to \infty$ is the expansion of the kernel $K^{-1}_{BB}(\psi,\psi')$. Its orbital momentum sum representation,

$$K^{-1}_{BB}(\psi,\psi') = \frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}l(\psi-\psi')}}{I_l(\omega R_B)K_l(\omega R_B)}, \tag{26}$$

follows from (19) by substituting $R_A \to R_B$. Now, if we would take the limit $R_B \to \infty$ in the last expression, using the asymptotic expansion of the Bessel functions,

$$I_\nu(z) \underset{z\to\infty}{\sim} \mathrm{e}^{z}, \qquad K_\nu(z) \underset{z\to\infty}{\sim} \mathrm{e}^{-z}, \tag{27}$$

we would get a diverging sum over l. Hence, arbitrarily high l contribute to the considered limit. Therefore, we have to take the uniform asymptotic expansion of the modified Bessel functions [25],

$$I_\nu(\nu z) \underset{\nu\to\infty}{\sim} \frac{1}{\sqrt{2\nu z}}\frac{\mathrm{e}^{\nu\eta(z)}}{(1+z^2)^{1/4}}(1+\cdots), \qquad K_\nu(\nu z) \underset{\nu\to\infty}{\sim} \sqrt{\frac{\pi}{2z}}\frac{\mathrm{e}^{-\nu\eta(z)}}{(1+z^2)^{1/4}}(1+\cdots)$$

(we do not need the explicit form of the function $\eta(z)$). With $\nu \to l$ and $z \to \omega R_B/l$ we get

$$I_l(\omega R_B)K_l(\omega R_B) = \frac{1}{2\sqrt{l^2+(\omega R_B)^2}}\left(1+\frac{t^2(1-t^2)(1-5t^2)}{8l^2}+\cdots\right) \tag{28}$$

with

$$t = \frac{l}{\sqrt{l^2+(\omega R_B)^2}}.$$

Further, in equation (13) we change the variable of integration ψ for z_2 according to

$$\psi = \arcsin\frac{z_2}{R_B} = \frac{z_2}{R_B} + \frac{1}{6}\left(\frac{z_2}{R_B}\right)^3 + \cdots \tag{29}$$

with

$$\mathrm{d}\psi = \frac{\mathrm{d}z_2}{R_B\sqrt{1-(z_2/R_B)^2}} = \frac{\mathrm{d}z_2}{R_B}\left(1+\frac{1}{2}\left(\frac{z_2}{R_B}\right)^2+\cdots\right). \tag{30}$$

The range $z_2 \in [-R_B, R_B]$ will extend to the whole axis in the following. We also make the corresponding substitution for the primed quantities in equation (13). In fact, this change of variables is the orthogonal projection of the right half of the circle corresponding to the section of the cylinder B onto the axis $z_1 = 0$. It is to be mentioned that the left half of that circle does not contribute to the Casimir force in the limit $R_B \to \infty$. This statement would not be true for any finite R_B; however, we are going to obtain an asymptotic expansion.

With these expansions, the kernel represented by equation (26) becomes a function of z_2 and z_2',

$$K_{BB}^{-1}(z,z') = \frac{1}{2\pi}\sum_{l=-\infty}^{\infty} 2\sqrt{l^2+(\omega R_B)^2} \times \left(1+\frac{1}{R_B^2}\left(\frac{\omega^2(\Gamma_p^2-5\omega^2)}{8\Gamma_p^6}+\mathrm{i}p((z_2)^3-(z_2')^3)\right)+\cdots\right)\mathrm{e}^{\mathrm{i}\frac{l}{R_B}(z_2-z_2')}. \qquad (31)$$

Finally, in the limit $R_B\to\infty$, the sum over the orbital momenta in (26) becomes an integral after

$$\frac{1}{R_B}\sum_{l=-\infty}^{\infty}\to\int_{-\infty}^{\infty}\mathrm{d}q, \qquad \frac{l}{R_B}\to q. \qquad (32)$$

In place of (26) we get

$$K_{BB}^{-1}(\psi,\psi') \underset{R_B\to\infty}{=} R_B^2\int_{-\infty}^{\infty}\mathrm{d}q\, 2\Gamma_q\,\mathrm{e}^{\mathrm{i}q(z_2-z_2')}\,H(q), \qquad (33)$$

where

$$\Gamma_q = \sqrt{\omega^2+q^2} \qquad (34)$$

and

$$H(q) = 1+\frac{1}{R_B^2}\left(\frac{\omega^2(\Gamma_q^2-5\omega^2)}{8\Gamma_q^6}+\mathrm{i}q\left(z_2^3-{z_2'}^3\right)\right)+\cdots. \qquad (35)$$

Now we are going to insert (33) into $N(\varphi,\varphi')$, equation (13), where we make the substitution (29). Further we use the momentum space representation

$$\Delta_\omega(\vec{z},\vec{z}') = \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{1}{2\Gamma_k}\,\mathrm{e}^{\mathrm{i}k(z_2-z_2')-\Gamma_k|z_1-z_1'|}, \qquad (36)$$

which can be obtained from (3) by carrying out the integration over k_1 and renaming k_2 as k. For $K_{AB}(\varphi,\psi)$ we use its definition (5), where we substitute $x^B(\psi)\to z$ with z_2 following from equation (29) and z_1 given by

$$z_1 = R_B\left(-1+\sqrt{1-\left(\frac{z_2}{R_B}\right)^2}\right) = -\frac{z_2^2}{2R_B}+\cdots. \qquad (37)$$

The result is

$$N(\varphi,\varphi') = \int\frac{\mathrm{d}z_2}{R_B\sqrt{1-\left(\frac{z_2}{R_B}\right)^2}}\int\frac{\mathrm{d}z_2'}{R_B\sqrt{1-\left(\frac{z_2'}{R_B}\right)^2}} \times\int\frac{\mathrm{d}k}{2\pi}\frac{\mathrm{e}^{\mathrm{i}k(x_2^A(\varphi)-z_2)-\Gamma_k|x_1^A(\varphi)-z_1|}}{2\Gamma_k}\int\frac{\mathrm{d}k'}{2\pi}\frac{\mathrm{e}^{\mathrm{i}k'(x_2^A(\varphi')-z_2')-\Gamma_{k'}|x_1^A(\varphi')-z_1'|}}{2\Gamma_k'} \times R_B^2\int\mathrm{d}q\,2\Gamma_q\,\mathrm{e}^{-q(z_2-z_2')}H(q) \qquad (38)$$

(here and in the following formulae all integrations go over the whole axis). In this expression, an expansion in $1/R_B$ can be made. Keeping contributions up to second order we get

$$N(\varphi,\varphi') = \int\mathrm{d}z_2\int\mathrm{d}z_2'\int\frac{\mathrm{d}k}{2\pi}\int\frac{\mathrm{d}k'}{2\pi}\int\mathrm{d}q\,\frac{\Gamma_q}{2\Gamma_k\Gamma_{k'}}\tilde{H} \times\mathrm{e}^{\mathrm{i}z_2(-k+q)+\mathrm{i}z_2'(k'-q)+\mathrm{i}kx_2^A(\varphi)-\Gamma_kx_1^A(\varphi)-\mathrm{i}k'x_2^A(\varphi')-\Gamma_{k'}x_1^A(\varphi')} \qquad (39)$$

with

$$\tilde{H} = 1 - \frac{1}{R_B}\frac{\left(\Gamma_k z_2^2 + \Gamma_{k'} {z'_2}^2\right)}{2} + \frac{1}{R_B^2}\left[\frac{\left(\Gamma_k z_2^2 + \Gamma_{k'} {z'_2}^2\right)^2}{8} - \frac{\left(z_2^2 + {z'_2}^2\right)}{2} + \frac{\omega^2(\Gamma_q^2 - 5\omega^2)}{8\Gamma q^6} + \mathrm{i}q\left(z_2^3 - {z'_2}^3\right)\right] + \cdots. \tag{40}$$

Here we used also $x_1^A(\varphi) - z_1 > 0$. In (39) the integrations over z_2 and z'_2 can be carried out delivering delta functions and their derivatives which allow us to remove two out of the three momentum integrations.

Let us first consider the zeroth-order term, $N^{(0)}_{\varphi,\varphi'}$, in the expansion which follows from (39) with $\tilde{H} \to 1$. It reads

$$N^{(0)}(\varphi,\varphi') = \int \mathrm{d}q \frac{1}{2\Gamma_q}\, \mathrm{e}^{\mathrm{i}q(x_2^A(\varphi) - x_2^A(\varphi')) - \Gamma_q(x_1^A(\varphi) + x_1^A(\varphi'))}. \tag{41}$$

The corresponding quantity in the orbital momentum basis (defined as in equation (9)) is

$$N^{(0)}_{l,l'} = (-1)^{l+l'} I_l(\omega R_A) I_{l'}(\omega R_A) \int \mathrm{d}q\, \frac{1}{2\Gamma_q}\left(\frac{\Gamma_q - q}{\Gamma_q + q}\right)^{\frac{l+l'}{2}} \mathrm{e}^{-2\Gamma_q L}, \tag{42}$$

where we used

$$\int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}\, \mathrm{e}^{-\mathrm{i}l\varphi + \mathrm{i}qx_2^A(\varphi) - \Gamma_q x_1^A(\varphi)} = (-1)^l I_l(\omega R_A)\, \mathrm{e}^{-\Gamma_q L}\left(\frac{\Gamma_q - q}{\Gamma_q + q}\right)^{\frac{l}{2}}. \tag{43}$$

The last expression can be derived from (17). The remaining integration can be carried out using equation (8.432 1) in [24] and results in

$$N^{(0)}_{l,l'} = (-1)^{l+l'} I_l(\omega R_A) K_{l+l'}(2\omega L) I_{l'}(\omega R_A), \tag{44}$$

in agreement with the corresponding formulae for a cylinder in front of a plane, see $A_{m,m'}$, equation (A7), in [16] or the corresponding formulae in [3]. Being inserted together with (19) into (10) with account for (11) the leading order in the limit $R_B \to \infty$ reproduces just the energy for a cylinder in front of a plane.

Now we consider the first next-to-leading order. It results from the $1/R_B$-contribution in $\tilde{H}$, equation (40), and its contribution to $N(\varphi,\varphi')$, equation (39), is

$$N^{(1)}(\varphi,\varphi') = \frac{-1}{2R_B}\int \mathrm{d}q\, \left(\frac{\partial}{\partial q}\, \mathrm{e}^{\mathrm{i}qx_2^A(\varphi) - \Gamma_q x_1^A(\varphi)}\right)\left(\frac{\partial}{\partial q}\, \mathrm{e}^{-\mathrm{i}qx_2^A(\varphi') - \Gamma_q x_1^A(\varphi')}\right). \tag{45}$$

In calculating this expression from (39) we represented z_2^2 by $(\partial/\partial k)(\partial/\partial q)$ and integrated by parts each derivative. The contribution from z'_2 appears to be the same.

Now we calculate the corresponding expression in the orbital momentum basis. The angular integrations can be carried out as before using (43). The corresponding integral in the angular momentum representation can be written in the form

$$N^{(1)}_{l,l'} = \frac{-1}{2R_B}\int \mathrm{d}q\, \left(\frac{\partial}{\partial q}\,(-1)^l I_l(\omega R_A)\left(\frac{\Gamma_q - q}{\Gamma_q + q}\right)^{\frac{l}{2}} \mathrm{e}^{-2L\Gamma_q}\right) \times \left(\frac{\partial}{\partial q}\,(-1)^{l'} I_{l'}(\omega R_A)\left(\frac{\Gamma_q - q}{\Gamma_q + q}\right)^{\frac{l'}{2}} \mathrm{e}^{-2L\Gamma_q}\right). \tag{46}$$

Simplifying this expression we obtain for $N^{(1)}_{l,l'}$ finally

$$N^{(1)}_{l,l'} = \frac{(-1)^{l+l'+1}}{2R_B} I_l(\omega R_A) I_{l'}(\omega R_A) \times \int \mathrm{d}q \, \frac{1}{\Gamma_q^2} \, (qL+l) \left(qL+l'\right) \left(\frac{\Gamma_q - q}{\Gamma_q + q}\right)^{\frac{l+l'}{2}} \mathrm{e}^{-2L\Gamma_q}. \quad (47)$$

Here the integration over q cannot be carried out as easily as in equation (42) and we keep it as is.

In this way, we obtained the expansion

$$N_{l,l'} = N^{(0)}_{l,l'} + N^{(1)}_{l,l'} + \cdots, \quad (48)$$

where $N^{(0)}_{l,l'}$, equation (44), is independent from R_B and $N^{(1)}_{l,l'}$, equation (46), is of order $1/R_B$. The dots denote the contributions of higher orders. Now we insert this expansion into equation (10) using (11) and (48),

$$E = \frac{1}{4\pi} \int_0^\infty \mathrm{d}\omega \, \omega \, \mathrm{Tr} \ln \left(\delta_{l,l'} - K^{-1}_{AA;l} \left(N^{(0)}_{l,l'} + N^{(1)}_{l,l'} + \cdots\right)\right), \quad (49)$$

and expand the logarithm,

$$\begin{aligned} E &= \frac{1}{4\pi} \int_0^\infty \mathrm{d}\omega \, \omega \, \mathrm{Tr} \ln \left(\delta_{l,l'} - K^{-1}_{AA;l} N^{(0)}_{l,l'}\right) \\ &\quad - \frac{1}{4\pi} \int_0^\infty \mathrm{d}\omega \, \omega \, \mathrm{Tr} \left(\delta_{l,l'} - K^{-1}_{AA;l} N^{(0)}_{l,l'}\right)^{-1} K^{-1}_{AA;l} N^{(1)}_{l',l''} + \cdots, \\ &\equiv E^{(0)} + E^{(1)} + \cdots. \end{aligned} \quad (50)$$

The leading order, $E^{(0)}$, is the energy for a cylinder in front of a plane and $E^{(1)}$ is the correction of order $1/R_B$. The latter can be rewritten in the form

$$E^{(1)} = \frac{-1}{4\pi} \int_0^\infty \mathrm{d}\omega \, \omega \, \mathrm{Tr} \left(\delta_{l,l'} K_{AA;l} - N^{(0)}_{l,l'}\right)^{-1} N^{(1)}_{l',l''}. \quad (51)$$

Equation (51) is the final step in the calculation of the $1/R_B$-correction to the vacuum energy for the radius of the cylinder B becoming large. It can be calculated numerically since all sums and integrations entering do converge.

In order to represent the result in a more instructive way, we represent the energy in terms of dimensionless functions,

$$E = \frac{1}{d^2} \, \tilde{E} \left(\frac{d}{R_B}, \frac{d}{R_A}\right), \quad (52)$$

where

$$d = L - R_A \quad (53)$$

is the separation between the two cylinders. For $R_B \to \infty$ we rewrite the last line in equation (50) in the form

$$E = \frac{1}{d^2} \left(\tilde{E}^{(0)} \left(\frac{d}{R_A}\right) + \frac{d}{R_B} \, \tilde{E}^{(1)} \left(\frac{d}{R_A}\right) + \cdots\right), \quad (54)$$

where $\tilde{E}^{(0)}(d/R_A)$ is a dimensionless function describing the case of the cylinder A in front of a plane and $\tilde{E}^{(1)}(d/R_A)$ describes the first correction for large R_B. Further we rewrite equation (54) in the form

$$E = \frac{1}{d^2} \, \tilde{E}^{(0)} \left(\frac{d}{R_A}\right) \left(1 + \frac{d}{R_B} \, \Delta E \left(\frac{d}{R_A}\right) + \cdots\right), \quad (55)$$

where $\Delta E \, (d/R_A) \equiv \tilde{E}^{(1)}(d/R_A)/\tilde{E}^{(0)}(d/R_A)$ is the relative correction.

The behavior of the function $\tilde{E}^{(0)}(d/R_A)$ is well known. For large argument, i.e., for $d \gg R_A$, it describes a small cylinder (or a cylinder at large separation) in front of a plane. It has a logarithmic behavior,

$$\tilde{E}^{(0)}\left(\frac{d}{R_A}\right) \sim \frac{-1}{16\pi \ln \frac{4d}{R_A}}, \tag{56}$$

which is due to the logarithmic behavior of the two-dimensional Green's function (3). For small argument, i.e., for $d \ll R_A$, its behavior follows from PFA,

$$\tilde{E}^{(0)}\left(\frac{d}{R_A}\right) \sim \frac{-\pi^3}{1920\sqrt{2}}\sqrt{\frac{R_A}{d}}\left(1 + \frac{7}{36}\frac{d}{R_A} + \cdots\right), \tag{57}$$

where we also included the first correction beyond PFA [16]. Numerical evaluations of this function can be found in [3, 22].

The function $\Delta E\,(d/R_A)$ can be calculated in a similar way. We start with large arguments and consider first the function $\tilde{E}^{(1)}(d/R_A)$. We expand for small R_A using

$$I_l\,(\omega R_A)\,K_l\,(\omega R_A) = \delta_{l,0}\left(-\ln\frac{\omega R_A}{2} - \gamma + \cdots\right) \tag{58}$$

and get from (46)

$$N^{(1)}_{l,l'} = \delta_{l,0}\delta_{l',0}\frac{-1}{2R_B}\int_{-\infty}^{\infty} \mathrm{d}q\,\frac{q^2L^2}{\Gamma^2}\,\mathrm{e}^{-2L\Gamma} + \cdots. \tag{59}$$

Here we took into account that we get in this approximation only the contribution from $\tilde{l} = \tilde{l}' = 0$ in $N^{(1)}_{l,l'}$, equation (46). Because of the explicit factor R_A in $D_{l,\tilde{l}}(q)$, equation (47), only $l = l' = 0$ did contribute. In this way we come to

$$\tilde{E}^{(1)}\left(\frac{d}{R_A}\right) \sim \frac{d}{8\pi}\int_0^\infty \mathrm{d}\omega\,\omega\int_0^\infty \mathrm{d}q\,\frac{q^2L^2}{\Gamma^2}\,\frac{\mathrm{e}^{-2L\Gamma}}{-\ln\frac{\omega R_A}{2} - \gamma}. \tag{60}$$

With the substitutions

$$q = \omega\sinh\theta, \qquad \omega = \frac{\sigma}{2L}, \tag{61}$$

and in leading order for small R_A and with (53) we get

$$\tilde{E}^{(1)}\left(\frac{d}{R_A}\right) \sim \frac{1}{64\pi\ln\frac{4d}{R_A}}\int_0^\infty \mathrm{d}\sigma\,\sigma^2\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{\sinh^2\theta}{\cosh\theta}\,\mathrm{e}^{-\sigma\cosh\theta}. \tag{62}$$

The integrations can be carried out resulting in

$$\tilde{E}^{(1)}\left(\frac{d}{R_A}\right) \sim \frac{1}{48\pi}\frac{1}{\ln\frac{4d}{R_A}}. \tag{63}$$

Together with (56) and (55) we get

$$\Delta\tilde{E}\left(\frac{d}{R_A}\right) = -\frac{1}{3} + O\left(\frac{R_A}{d}\right), \tag{64}$$

which is the relative correction at large separation.

Now we consider the opposite limit of small argument. It corresponds to a large cylinder A in close separation from the larger cylinder B. In this case, the energy can be calculated from PFA. In general, if both cylinders are large, the energy $E_{A,B}$ is given in PFA by

$$E_{A,B} = -\frac{\pi^3}{1920\sqrt{2}}\frac{1}{d^2}\sqrt{\frac{\bar{R}}{d}}, \tag{65}$$

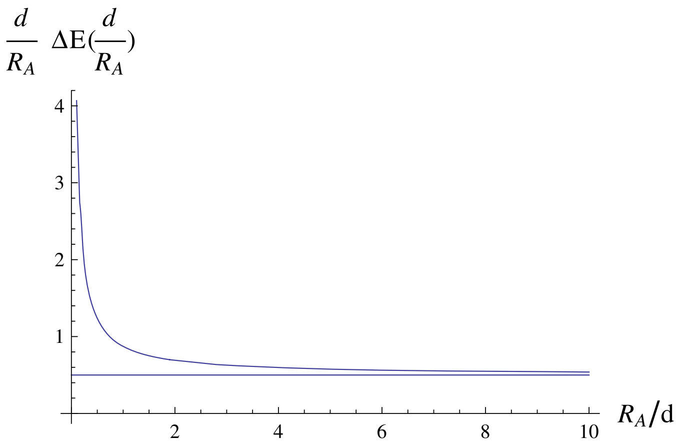

Figure 3. The function $d/R_A\,\Delta E(d/R_A)$ and its PFA-limit $1/2$.

where

$$\bar{R} = \frac{R_A R_B}{R_A + R_B} \tag{66}$$

is the reduced radius. It can be expanded for $R_B \gg R_A$,

$$\bar{R} = R_A - \frac{R_A^2}{R_B} + \cdots. \tag{67}$$

So we get for E, equation (50), for $d \ll R_A \ll R_B$, the following approximation,

$$E = -\frac{\pi^3}{1920\sqrt{2}}\frac{1}{d^2}\sqrt{\frac{R_A}{d}}\left(1 + \frac{7}{36}\frac{d}{R_A} - \frac{1}{2}\frac{R_A}{R_B} + \cdots\right), \tag{68}$$

where we also added the correction beyond PFA which may be of the same order as the correction for large R_B. Comparing (55) with (68) we infer the behavior for small argument,

$$\Delta\tilde{E}\left(\frac{d}{R_A}\right) = -\frac{1}{2}\frac{R_A}{d} + O\left(1\right). \tag{69}$$

As already mentioned, this function can be evaluated numerically using equation (51). This evaluation was carried out with a truncation of the orbital momenta sums which ensures a precision of at least 2 digits. Surprisingly, the numerical results can be described remarkably well by the fit

$$\Delta E^{\text{fit}}\left(\frac{d}{R_A}\right) = 0.359 + 0.514\,\frac{R_A}{d} - 0.002\left(\frac{R_A}{d}\right)^2 + 0.0001\left(\frac{R_A}{d}\right)^3. \tag{70}$$

In figure 3, it is shown how the function $d/R_A\,\Delta E(d/R_A)$ approaches the value $1/2$ which follows from the PFA, equation (69).

Now we consider the configuration of a cylinder A inside the cylinder B; see figure 2. The calculation goes in close parallel to the former case, and we indicate only the necessary changes. First of all, we have to define the coordinates parameterizing the cylinder B now. These were given by (4). Whereas $\vec{x}^A$ does not change, we have for $\vec{x}^B$ now

$$B: \quad \vec{x}^B(\varphi) = \begin{pmatrix} R_B(1-\cos\varphi) \\ R_B\sin\varphi \end{pmatrix}, \tag{71}$$

which appears after a reflection on the plane $(x_1 = 0)$.

The expression (26) for $K_{BB}^{-1}(\psi, \psi')$ does not change. We can keep the substitution (29) and (30), whereas in place of (37) we have now

$$z_1 = R_B \left(1 - \sqrt{1 - \left(\frac{z_2}{R_B}\right)^2}\right) = -\frac{z_2^2}{2R_B} + \cdots . \tag{72}$$

The first term of the expansion has the opposite sign as compared to (37). In fact, this is the only change we have to account for in the subsequent formulae. In these we have first to consider $K_{BB}^{-1}(z, z')$, equation (31). It remains unchanged. Next is $N(\varphi, \varphi')$, equation (38). Here the variables z_1 and z_1' appear only in the exponential together with Γ_k and $\Gamma_{k'}$. In making in (38) the expansion in $1/R_B$, the sign change appears in $\tilde{H}$, equation (40), just in the first-order contribution. All remaining calculations go in the same way as before. In this way, the changed sign can be traced until the final formula for the energy, equation (64), which in the case of an inscribed cylinder reads

$$E = \frac{-1}{16\pi L^2} \frac{1}{\ln \frac{4L}{R_A}} \left(1 + \frac{L}{3R_B} + \cdots\right). \tag{73}$$

In this case the energy is increased, again in correspondence with expectations.

4. Conclusions

In the previous section, we considered the vacuum energy of a scalar field obeying Dirichlet boundary conditions on two cylinders and calculated the asymptotic expansion of this energy for one of the cylinders becoming large. We have shown how to construct this expansion and wrote down the first two orders in a general form. As a particular example, we considered the first order in the special case when the separations between the cylinders become large, equation (64). The other case, when the separation becomes small is covered by PFA and resulted in equation (69).

The asymptotic expansion for large radius R_B involves arbitrarily high orbital momenta for the cylinder B. This is similar to the expansion for small separation, but in detail, of course, different. It should be mentioned that the limit of one cylinder becoming large cannot be obtained from PFA unlike the case of small separation. In this sense, it is an independent calculation. However, it should be related to a perturbative expansion which emerges if considering the large cylinder as a small deviation from a plane. For consistency reasons, it would be interesting to check this.

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References

[1] Bordag M, Klimchitskaya G L, Mohideen U and Mostepanenko V M 2009 *Advances in the Casimir Effect* (Oxford: Oxford University Press)

[2] Bulgac A, Magierski P and Wirzba A 2006 Scalar Casimir effect between Dirichlet spheres or a plate and a sphere *Phys. Rev.* D **73** 025007

[3] Emig T, Jaffe R L, Kardar M and Scardicchio A 2006 Casimir interaction between a plate and a cylinder *Phys. Rev. Lett.* **96** 080403

[4] Kenneth O and Klich I 2006 Opposites attract—a theorem about the Casimir force *Phys. Rev. Lett.* **97** 160401
[5] Renne M J 1971 Microscpic theory of retarded van der Waals forces between macroscopic dielectric bodies *Physica* **56** 125–37
[6] Bordag M, Robaschik D and Wieczorek E 1985 Quantum field theoretic treatment of the Casimir effect *Ann. Phys.* **165** 192
[7] Li H and Kardar M 1992 Fluctuation-induced forces between manifolds immersed in correlated fluids *Phys. Rev.* A **46** 6490–500
[8] Milton K A, Parashar P and Wagner J 2008 From multiple scattering to van der Waals interactions: exact results for eccentric cylinders
[9] Milton K A and Wagner J 2008 Exact Casimir interaction between semitransparent spheres and cylinders *Phys. Rev.* D **77** 045005
[10] Cavero-Pelaez I, Milton K A and Kirsten K 2007 Local and global Casimir energies for a semitransparent cylindrical shell *J. Phys. A: Math. Theor.* **40** 3607–32
[11] Romeo A and Milton K A 2005 Casimir energy for a purely dielectric cylinder by the mode summation method *Phys. Lett.* B **621** 309–17
[12] Rahi T, Emig T, Jaffe R and Kardar M 2008 Nonmonotonic effects of parallel sidewalls on Casimir forces between cylinders *Phys. Rev.* A **78** 012104
[13] Rahi T, Rodriguez A, Emig T, Jaffe R, Johnson S and Kardar M 2008 Nonmonotonic effects of parallel sidewalls on Casimir forces between cylinders *Phys. Rev.* A **77** 030101
[14] Mazzitelli F D, Dalvit F C and a dn Lombardo D R A 2006 Exact zero-point interaction energy between cylinders *New J. Phys.* **8** 240
[15] Dalvit D A R, Lombardo F C, Mazzitelli F D and Onofrio R 2006 Exact Casimir interaction between eccentric cylinders *Phys. Rev.* A **74** 020101
[16] Bordag M 2006 The Casimir effect for a sphere and a cylinder in front of plane and corrections to the proximity force theorem *Phys. Rev.* D **73** 125018
[17] Bordag M and Nikolaev V 2008 Casimir force for a sphere in front of a plane beyond proximity force approximation *J. Phys. A: Math. Theor.* **41** 164001
[18] Gies H, Langfeld K and Moyaerts L 2003 Casimir effect on the worldline *J. High Energy Phys.* JHEP06(2003)018
[19] Gies H and Klingmuller K 2006 Quantum energies with worldline numerics *J. Phys. A: Math. Gen.* **39** 6415–22
[20] Gies H and Klingmueller K 2006 Worldline algorithms for Casimir configurations *Phys. Rev.* D **74** 045002
[21] Emig T 2008 Fluctuation-induced quantum interactions between compact objects and a plane mirror *J. Stat. Mech.* **08** P04007
[22] Lombardo F C, Mazzitelli F D and Villar P I 2008 Numerical evaluation of the Casimir interaction between cylinders *Phys. Rev.* D **78** 085009
[23] Brown-Hayes M, Dalvit D A R, Mazzitelli F D, Kim W-J and Onofrio R 2005 Towards a precision measurement of the Casimir force in a cylinder plane geometry *Phys. Rev.* A **72** 052102
[24] Gradshteyn I S and Ryzhik I M 1994 *Table of Integrals, Series and Products* (New York: Academic)
[25] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions: with Formulas, Graphs and Mathematical Tables* (New York: Dover)